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LETTER TO THE EDITOR

Path integrals, disordered systems, and some ensuing mathematical questions

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Abstract. The path integral describing a particle in a randomly distributed potential is examined. A heuristic discussion is given, which should elucidate the qualitative features as well as the mathematical problems of density-of-states calculations.

1. Introduction

A standard approach to the problem of density of states in the presence of randomly distributed potentials (Edwards and Gulyaev 1964) depends on first expressing the Green function as a path integral. It is then customary to approximate this integral by the second cumulant (cf Kubo 1962), so as to yield, after an obvious change of variables,

$$(2\pi it)^{-s/2}\int_{\eta(0)=\eta(t)=0}\mathscr{D}(\eta)\exp\left(\tfrac{1}{2}it^{-1}\int_0^1 d\tau\dot{\eta}^2-\tfrac{1}{2}\rho t^2\iint_0^1 d\tau_1\, d\tau_2 W(\eta(\tau_1)-\eta(\tau_2))\right)$$
$$= G(t;0,0) \underset{\mathrm{df}}{=} G(t). \tag{1}$$

Here the mass is unity, s is the number of spatial dimensions, ρ is the density of the potentials, and the integral is normalized so as to yield unity when $\rho = 0$. Further, W is the correlation function of the potential V,

$$W(\eta_1-\eta_2) = \int d^sR\, V(\eta_1-R)V(\eta_2-R). \tag{2}$$

The density of states is now given by

$$n(E) = \pi^{-1}\lim_{\epsilon\downarrow 0}\mathrm{Re}\int_0^\infty dt G(t)\, e^{it(E+i\epsilon)} \tag{3}$$

If the last integral diverges, the finite part is to be taken.

In most of the studies of the problem (eg Bezák 1970, Samathiyakanit 1974, and references given therein) one proceeds by assuming an explicit expression for V and hence for W, and then by evaluating the path integral in (1), which often requires further

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approximations. Expressions like the following were found for the asymptotic limits as $E \to \pm\infty$:

$$n(E) \sim a_1 E^{1/2} \qquad \text{as } E \to \infty, \tag{4a}$$

$$n(E) \sim a_2 |E|^{-1} \exp(-a_3 E^2) \qquad \text{as } E \to -\infty, \tag{4b}$$

where the a_j are positive constants.

However, experience with path integrals has been accumulating, with respect to both manipulation and rigour (eg McLaughlin 1972a, Tarski 1975). It appears therefore natural to try to extract some information directly from equations (1) and (3). Unevaluated path integrals, in fact, may well have a greater heuristic value than their approximate evaluations. (Unfortunately, a rigorous discussion of (1) could now be given only for a very limited class of functions W.)

We discuss here several aspects of the above path integral, of G, and of n. Our hope is to make the calculations in cited works more intuitive, and the following points can also be considered as conjectures, or as problems for a systematic study.

2. Analyticity and asymptotics

Analyticity is known to have far-reaching consequences. Since the path integral is characterized by the weight factor $\exp(\frac{1}{2}\,\mathrm{i}t^{-1}\int \mathrm{d}\tau\dot{\eta}^2)$, we conjecture analyticity in t when $\operatorname{Im} t < 0$, for sufficiently regular functions W. Such analyticity is basic for what follows and could perhaps be more easily established by independent methods. Moreover, the manifest singularity of G at $t = 0$ is not expected to be cancelled by the path integral.

Let us now assume that for $t \simeq 0$, the integral in (1) is well approximated by setting $\eta(\tau) = \eta(0) = 0$ in the integrand. Thus for $s = 3$,

$$G(t) \simeq (2\pi \mathrm{i}t)^{-3/2} \exp(-\tfrac{1}{2}\rho t^2 W(0)). \tag{5}$$

Then the method of stationary phase (eg Erdélyi 1956) suggests that the asymptotic behaviour of $n(E)$ as $E \to \pm\infty$ is determined by $G(t)$ near $t = 0$, and one obtains a conclusion similar to (4a),

$$n(E) \lesssim (\text{constant})|E|^{1/2} \qquad \text{as } |E| \to \infty. \tag{6}$$

(An infinite part was eliminated here).

The fact that for $E \to -\infty$ one finds $n(E) < c|E|^{1/2}$ rather than $n(E) \sim c|E|^{1/2}$ can be understood as follows. For $E < 0$ one can displace the contour to $\operatorname{Im} t < 0$, where G is (presumably) analytic, and then a steepest-descent calculation can give a more precise estimate. One does not have this possibility with $E > 0$, $\operatorname{Im} t > 0$.

These considerations apply directly to the case of nearly-flat potentials. Let us suppose that in some approximation one can set $W \simeq$ constant $\underset{\mathrm{df}}{=} \langle W \rangle$, so that

$$G(t) \simeq (2\pi \mathrm{i}t)^{-s/2} \exp(-\tfrac{1}{2}\rho t^2 \langle W \rangle). \tag{7}$$

This relation should apply for all $t \geqslant 0$, in contrast to the situation in (5). Then the Fourier transform (3) can be expressed in terms of parabolic cylinder functions (Bezák 1970), and the combination of asymptotic methods as above leads to the relations (4).

We make two further remarks. First, analytic continuation of path integrals in the time parameter was also considered in connection with problems of barrier penetration by McLaughlin (1972b). Second, if we continue (1) analytically to t negative imaginary, we obtain a Wiener integral. Such integrals usually are a more versatile tool than the path integral of Feynman. However, in this case the Wiener integral would not give information about $n(E)$ as $E \to \infty$.

3. Use of Fourier development

The conditions $\eta(0) = \eta(1) = 0$ in (1) suggest that we exploit the sine series,

$$\eta(\tau) = \sum_j c_j(2^{1/2}/j\pi)\sin j\pi\tau, \tag{8a}$$

so that

$$\int_0^1 \mathrm{d}\tau\dot{\eta}(\tau)^2 = \sum_j c_j^2. \tag{8b}$$

A functional $F(\eta(\cdot))$ in the integrand is then transformed to

$$\bar{F}(c_1, c_2, \ldots) \underset{\mathrm{df}}{=} F\left(\sum_j c_j(2^{1/2}/j\pi)\sin j\pi\cdot\right). \tag{8c}$$

Following the work of Friedrichs and Shapiro (1957), we conjecture that for reasonable integrands F,

$$\int \mathscr{D}(\eta)\exp\left(\frac{\mathrm{i}}{2\sigma}\int_0^1 \mathrm{d}\tau\dot{\eta}^2\right)F(\eta)$$
$$= \lim_{N\to\infty}\int\ldots\int_{-\infty}^{\infty}\prod_{j=1}^{N}\left(\frac{\mathrm{d}c_j}{(2\pi\sigma\mathrm{i})^{1/2}}\mathrm{e}^{\mathrm{i}c_j^2/2\sigma}\right)\bar{F}(c_1,\ldots,c_N,0,0,\ldots). \tag{9}$$

To illustrate such manipulations, let us consider the δ potential in one dimension, for which $n(E)$ has the exponential decrease of the form $\exp(-a|E|^{3/2})$ as $E \to -\infty$ (rather than e^{-bE^2}). This problem has been studied extensively from several points of view, but the analysis of the corresponding path integral in (1) remains a challenge.

This integral remains also very non-trivial in the approximation $N = 1$ of (9), where

$$G^{(N=1)}(t) = (2\pi\mathrm{i}t)^{-1}\int_{-\infty}^{\infty}\mathrm{d}c\,\mathrm{e}^{\mathrm{i}c^2/2t}\exp\left(-\tfrac{1}{2}\rho t^2\iint_0^1 \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\delta(c'\sin\pi\tau_1 - c'\sin\pi\tau_2)\right). \tag{10}$$

Here $c' = c2^{1/2}/\pi$, the last exponent reduces to

$$\exp(-\rho t^2|c|^{-1}K), \qquad K = \text{constant} > 0, \tag{11}$$

and then

$$n^{(N=1)}(E) = \pi^{-1}\lim_{\epsilon\downarrow 0}\mathrm{Re}\int_0^\infty \frac{\mathrm{d}t}{2\pi t\mathrm{i}}\mathrm{e}^{\mathrm{i}t(E+\mathrm{i}\epsilon)}\int_{-\infty}^{\infty}\mathrm{d}c\,\mathrm{e}^{(\mathrm{i}c^2/2t)-\rho t^2|c|^{-1}K}. \tag{12}$$

This expression does not seem easy to estimate. We note only the following: by making the change of variable $c \to ct^2$ we find this t integral,

$$\int_0^\infty \mathrm{d}tt\,\mathrm{e}^{\mathrm{i}t(E+\mathrm{i}\epsilon)}\,\mathrm{e}^{\frac{1}{2}\mathrm{i}t^3c^2}. \tag{13}$$

This integral resembles that for Airy functions, which show a slower than Gaussian decrease.

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